RO(G)-graded norms for de Rham-Witt forms

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Your talk will begin after this brief advertisement

Introduction

deep connection between p-adic cohomology and THH (Hesselholt-Madsen, Bhatt-Morrow-Scholze, many others)

for most of human history: THH studied via **genuine** equivariant homotopy theory

Nikolaus-Scholze 2017: reformulate cyclotomic spectra in terms of **naive** homotopy theory

my research programme: put genuine equivariance back in

Why genuine equivariance?

$$\pi_* X^G \to \pi_* X^{hG}$$

is generally a kind of **decompletion** ; used for this purpose in upcoming work of Devalapurkar-Hahn-Raksit-Yuan

But, we can do much more.

- RO(G)-graded homotopy
- Tambara functors (multiplicative norm maps)
- equivariant slice filtration (after Hill-Hopkins-Ravenel)

overarching goal since 2018: modify Bhatt-Morrow-Scholze construction to use slice filtration on TC^- and TP instead of Postnikov filtration

this is now doable!

Goals today

- compute $\mathrm{TC}^-_\bigstar(S; \mathbf{Z}_p)$ and $\mathrm{TP}_\bigstar(S; \mathbf{Z}_p)$ when S is quasiregular semiperfectoid
- compute (a large portion of) $\mathrm{TR}^n_\bigstar(S; \mathbf{Z}_p)$ when S is smooth over a perfectoid ring
- **main application:** extend Angeltveit's norm map of Witt vectors to the de Rham-Witt complex

General takeaways

- canonical identifications instead of generators and relations
- index representations by $\{\dim \alpha^H\}_{H \leq G}$ instead of their irreducible decomposition
- negative indexing to avoid casework
- spectral sequence $\pi_\bigstar((X^{\Phi H})_{hG/H}) \Rightarrow \pi_\bigstar X^G$

Related papers

- Angeltveit, The norm map of Witt vectors
- Molokov, Prismatic cohomology and de Rham-Witt forms
- Devalapurkar-Misterka, Generalized n-series and de Rham complexes

$\sim$

- S., Prisms and Tambara functors I
- Mao, Prismatic logarithm and prismatic Hochschild homology via norm
- Bouis-Gazda, The cyclosyntomic regulator of a number field

$\sim$

- S., RO(G)-*graded norms for prismatic and de Rham-Witt forms* (in preparation)

Recollection on Witt vectors

S commutative ring $\rightsquigarrow$ ring of (p-typical) Witt vectors

$$\mathrm{W}(S) = \varprojlim_{n,R} \mathrm{W}_n(S)$$

Notation: $\mathrm{W}_0(S) = S$, $\mathrm{W}_n(\mathbf{F}_p) = \mathbf{Z}/p^{n+1}$, $\mathrm{W}(\mathbf{F}_p) = \mathbf{Z}_p$. Later will be convenient to set $\mathrm{W}_n(S) = 0$ for $n < 0$. **As a set,**

$\mathrm{W}_n(S) = S^{\times n+1}$; ring structure such that

$$\text{👻}\colon \mathrm{W}_n(S) \to S^{\times n+1}$$

$$\begin{aligned} w_0 &= a_0 \\ w_1 &= a_0^p + pa_1 \\ w_2 &= a_0^{p^2} + pa_1^p + p^2 a_2 \\ &\dots \end{aligned}$$

is a ring map

Operations on Witt vectors

$$\begin{tikzcd} \mathrm{W}(S) \arrow[r, shift left, "F"] \arrow[d, two heads] & \mathrm{W}(S) \arrow[l, shift left, "V"] \arrow[d, two heads] \\ \mathrm{W}_n(S) \arrow[r, shift left=2, "F"] \arrow[r, shift right=2, "R"'] & \mathrm{W}_{n-1}(S) \arrow[l, "V" description] \end{tikzcd}$$

Frobenius, Verschiebung, Restriction

$$F(w_0, \ldots, w_n) = (w_1, \ldots, w_n)$$
$$V(w_0, \ldots, w_{n-1}) = (0, pw_0, \ldots, pw_{n-1})$$
$$R(w_0, \ldots, w_n) = (w_0, \ldots, w_{n-1})$$

$$FV = p$$

Operations on Witt vectors (cont.)

$$
\begin{tikzcd}
\mathrm{W}(S) \arrow[r, shift left, "F"] \arrow[d, two heads] & \mathrm{W}(S) \arrow[l, shift left, "V"] \arrow[l, bend left, "N"] \arrow[d, two heads] \\
\mathrm{W}_n(S) \arrow[r, shift left=2, "F"] \arrow[r, shift right=2, "R"'] & \mathrm{W}_{n-1}(S) \arrow[l, "V" description] \arrow[l, bend left=40, "N"]
\end{tikzcd}
$$

$$F(w_0, \ldots, w_n) = (w_1, \ldots, w_n)$$
$$V(w_0, \ldots, w_{n-1}) = (0, pw_0, \ldots, pw_{n-1})$$
$$R(w_0, \ldots, w_n) = (w_0, \ldots, w_{n-1})$$

Angeltveit: $\exists$ a *N*orm map

$$FVx = px$$
$$FNx = x^p$$

comes from **Tambara structure** on THH

What is the Norm?

$$FNx = x^p$$
$$N(w_0, \dots, w_{n-1}) = (?, w_0^p, \dots, w_{n-1}^p)$$

Definition

A δ-**structure** on a ring A is a function $\delta\colon A \to A$ such that

$$\phi(f) = f^p + p\delta(f)$$

is a ring map.

Fact: $\mathrm{W}(S)$ is a δ-ring, in fact the cofree δ-ring on S, with $\phi = F$.

Borger's formula for the Norm

$$\begin{aligned} FNx &= x^p \\ &= Fx - p\delta(x) \\ Nx &= x - pF^{-1}\delta(x) \\ &= x - V\delta(x) \end{aligned}$$

Determined in ghost coordinates by

$$FNx = x^p$$
$$N \equiv \mathrm{id} \bmod V$$
$$N(w_0, \dots, w_{n-1}) = (w_0, w_0^p, \dots, w_{n-1}^p)$$

Witt coordinates:

$$(a_0, \dots, a_n) = N^n a_0 + \cdots + V^k N^{n-k} a_k + \cdots + V^n a_n$$

Expresses that $\mathrm{W}_\bullet(S)$ is the free $\mathbf{Z}_p$-Tambara functor on $S^{\mathrm{triv}} \in \mathrm{CRing}^{B\mathbf{Z}_p}$ [Brun]

Iterates of the Norm (S.)

To study iterates of the Norm, we need more operations!

$$\phi(f) = f^p + p\delta_1(f) \qquad \phi(f) = f^p + p\theta_1(f)$$

$$\phi^2(f) = f^{p^2} + p\delta_1(f)^p + p^2\delta_2(f) \qquad \phi(f^p) = f^{p^2} + p^2\theta_2(f)$$

$$\phi^n(f) = \sum_{k=0}^{n} p^k\delta_k(f)^{p^{n-k}} \qquad \phi(f^{p^{n-1}}) = f^{p^n} + p^n\theta_n(f)$$

Different perspective:

$$\begin{aligned}\phi^2(f) &= \phi(f^p + p\theta_1(f)) \\ &= f^{p^2} + p\theta_1(f)^\phi + p^2\theta_2(f)\end{aligned}$$

$$\phi^n(f) = f^{p^n} + \sum_{k=1}^{n} p^k\theta_k(f)^{\phi^{n-k}}$$

Formula for iterates (S.)

Explicit

$$N = \mathrm{id} - V\theta_1$$
$$N^2 = \mathrm{id} - V\theta_1 - V^2\theta_2$$
$$N^n = \mathrm{id} - \sum_{k=1}^{n} V^k \theta_k(f)$$

Implicit

$$\mathrm{id} = N\delta_0 + V\delta_1$$
$$\mathrm{id} = N^2\delta_0 + VN\delta_1 + V^2\delta_2$$
$$\mathrm{id} = \sum_{k=0}^{n} V^k N^{n-k} \delta_k$$

"Witt coordinates" decomposition of the identity

de Rham-Witt complex

S smooth R-algebra $\rightsquigarrow$ **relative de Rham-Witt complex** $\mathrm{W}_\bullet\Omega^*_{S/R}$ [Langer-Zink]

Focus on the case where R is perfectoid (probably not necessary, but easiest to relate to topology).

technically $\mathrm{W}_\bullet\Omega^*_{\mathrm{Spf}\, S/\,\mathrm{Spf}\, R} := \varprojlim_n \mathrm{W}_\bullet\Omega^*_{(S/p^n)/(R/p^n)}$

Goal

Using Tambara structure of TR, extend N from $\mathrm{W}(S)$ to $\mathrm{W}\Omega^*_{S/R}$.

Problem

This is not how equivariant homotopy theory works!

RO(G)-graded TR

$\mathrm{TR}^n := \mathrm{THH}^{C_{p^n}}$ is naturally graded on $\mathrm{RO}(\mathbf{Z}/p^n)$

$$\mathrm{RO}(\mathbf{Z}/p^n) \underset{R}{\overset{F}{\rightrightarrows}} \mathrm{RO}(\mathbf{Z}/p^{n-1})$$

$$F(\alpha) = \mathrm{res}^{\mathbf{Z}/p^n}_{p\mathbf{Z}/p^n\mathbf{Z}}\,\alpha \qquad R(\alpha) = \alpha^{p^{n-1}\mathbf{Z}/p^n\mathbf{Z}}$$

$$\mathrm{TR}^n_\alpha \underset{V}{\overset{F}{\rightleftarrows}} \mathrm{TR}^{n-1}_{F(\alpha)} \qquad \mathrm{TR}^n_\alpha \xrightarrow{R} \mathrm{TR}^{n-1}_{R(\alpha)}$$

(so TR graded on $\mathrm{RO}(\mathbf{Z}_p)^\wedge := \varprojlim_{n,R} \mathrm{RO}(\mathbf{Z}/p^n)$)

$$\mathrm{TR}^{n-1}_\alpha \xrightarrow{N} \mathrm{TR}^n_{\mathrm{ind}\,\alpha}$$

$\alpha \in \mathbf{Z} \implies F(\alpha), R(\alpha) \in \mathbf{Z}$; but $\mathrm{ind}(\alpha) \in \mathbf{Z} \iff \alpha = 0$

Ingredients

Definition

The **canonical filtration** on the de Rham-Witt complex is

$$\begin{aligned}\mathrm{Fil}^{k+1} W_n\Omega^* &= \ker(W_n\Omega^* \twoheadrightarrow W_k\Omega^*) \\ &= V^k + dV^k\end{aligned}$$

$\mathrm{Fil}^k W_n\Omega^* = W_n\Omega^*$ for $k \leq 0$

Definition

The **Adams operations** on the de Rham complex are given by

$$\psi^k(\omega \in \Omega^j) = k^j \omega$$

de Rham-Witt norms

Theorem (S.)

Let S be formally smooth over a perfectoid ring R. There are norm maps

$$N\colon \mathrm{W}_n\Omega^j_{S/R} \to \mathrm{W}_{n+1}\Omega^j_{S/R} \otimes (\cdots) + \operatorname{Fil}^1 \mathrm{W}_{n+1}\Omega^{pj}_{S/R} \otimes (\cdots)$$

satisfying

$$FN\omega = \omega^p, \qquad N \equiv \psi^p \bmod \operatorname{Fil}^1$$

and extending Angeltveit's norm maps for $j = 0$.

- for $j > 0$, $+$ is $\oplus$
- $\otimes(\cdots)$ is tensoring over $\mathrm{W}_{n+1}(R)$ with a free $\mathrm{W}_{n+1}(R)$-module of rank one. Doesn't change $+, F, V$; but changes R and multiplicative structure. Roughly says that $N(\omega) =$ "$\omega^{\mathbf{Z}/p}$" is divisible by "$(\mathbf{Z}/p)!$"

A formula for $j > 0$

$$N\colon \mathrm{W}_n\Omega^j_{S/R} \to \mathrm{W}_{n+1}\Omega^j_{S/R} + \mathrm{Fil}^1\,\mathrm{W}_{n+1}\Omega^{pj}_{S/R}$$

$$FN\omega = \omega^p, \qquad N\omega \equiv \psi^p(\omega) \bmod \mathrm{Fil}^1$$

for $j > 0$,

$$N(\omega) = (p - V(1))\frac{\psi^p(\omega)}{p} + V\frac{\omega^p}{p}$$

- easy to see this satisfies the identities
- makes sense because for $j > 0$, $\psi^p(\omega)$ and ω^p are canonically divisible by p
- ($\Omega^*_{S/R} = \mathrm{HH}_*(S/R)$ has divided powers since $\mathrm{HH}(S/R)$ is an animated commutative ring)

A unified formula

$$N(\omega) = (p - V(1))\frac{\psi^p(\omega)}{p} + V\frac{\omega^p}{p}$$
$$= \psi^p(\omega) - \left[\frac{V(1)\psi^p(\omega) - V\omega^p}{p}\right]$$

using $V(1)x = VFx$,

$$= \psi^p(\omega) - V\left[\frac{F\psi^p(\omega) - \omega^p}{p}\right]$$

Define

$$\psi^k(\omega \in \mathrm{W}\Omega^j) = k^j\omega \qquad \delta(\omega) = \frac{F\psi^p(\omega) - \omega^p}{p}$$

then

$$N(\omega) = \psi^p(\omega) - V\delta(\omega)$$

A unified formula

Can similarly extend δ_k and θ_k to $\mathrm{W}\Omega^*$

$$N^k : \mathrm{W}_n\Omega^j_{S/R} \to \sum_{r=0}^{k} \mathrm{Fil}^r\,\mathrm{W}_{n+k}\Omega^{p^r j}_{S/R} \otimes (\cdots)$$

Implicit: "Witt coordinates" decomposition of Adams operations

$$\psi^{p^k} = \sum_{r=0}^{k} V^r N^{k-r} \delta_r$$

Explicit:

$$N^k(\omega) = \omega^{p^k} - \sum_{r=1}^{k} V^k \theta_k(\omega)$$

Examples

Example

If $\omega \in W\Omega^1$, then $N(\omega) = (p - V(1))\omega$.

Example

In char p, $N(\omega) = V\frac{\omega^p}{p}$. Not obvious since the factor $(p - V(1))$ is partially absorbed by the ideal twist; but still turns out to be true.

Example

Let k be a perfect $\mathbf{F}_p$-algebra, $S = k[x_1, y_1, \ldots, x_p, y_p]$, and

$$\tau = \sum_{i=1}^{p} dx_i\, dy_i \in W\Omega^2_S \qquad \omega = \prod_{i=1}^{p} dx_i\, dy_i \in W\Omega^{2p}_S.$$

Then $\tau^p = p!\omega$, so $N(\tau) = V(p-1)!\omega$.

Strategy

pioneering work on $\mathrm{TR}_\bigstar$ done by Angeltveit-Gerhardt
introduced "homotopy orbits to TR" spectral sequence (HOTRSS)

$$\pi_\bigstar\left[(\mathrm{THH}^{\Phi p^s\mathbf{Z}/p^n\mathbf{Z}})_{h\mathbf{Z}/p^s}\right] \Rightarrow \mathrm{TR}^n_\bigstar$$

$\Rightarrow$ gave algorithm to compute $\mathrm{TR}^n_\bigstar(\mathbf{F}_p)$ (and also $\mathrm{TR}^n_\bigstar(\mathbf{Z};\mathbf{F}_p)$ and $\mathrm{TR}^n_\bigstar(\ell;V(1))$) as ab. gps.

Need several extensions:

- compute Mackey + multiplicative structure
- generalize to perfectoid rings
- generalize to smooth algebras
- get closed-form answers

Will use BMS to understand the orbits

Prisms

Definition

A **prism** is a pair (A, I) where

- A is a δ-ring
- I is an ideal locally gen'd by a non-zero divisor
- A is (derived) (p, I)-cplt
- $p \in I + \phi(I)$ ("prism condition")

A prism is

- **orientable** if $I = (\widetilde{\xi})$ (safe to always assume)
- **transversal** if A/I is p-torsionfree

every prism admits a surjection from a transversal one

$$I_n := I\phi(I) \cdots \phi^{n-1}(I)$$

Examples

If R is "quasiregular semiperfectoid" (qrsp), its **absolute prismatic cohomology** Δ_R is a prism with **Nygaard filtration**

$$\mathcal{N}^{\geq i}\Delta_R = \phi^{-1}(I^{(\!(i)\!)})$$

where $(\!(i)\!) := \max(i, 0)$.

Example (Perfect $\mathbf{F}_p$-algebras)

Let k be a perfect $\mathbf{F}_p$-algebra. Then $\Delta_k = \mathrm{W}(k)$ with $\widetilde{\xi} = p$. We have $I_n = p^n\mathrm{W}(k)$ and $\mathcal{N}^{\geq i} = p^{(\!(i)\!)}\mathrm{W}(k)$.

Example (Perfectoid rings)

Let R be a perfectoid ring. Then $\Delta_R = \mathrm{A}_{\mathrm{inf}}(R) := \varprojlim_{n,F} \mathrm{W}_n(R)$, with orientation $\widetilde{\xi}$ depending on R. The Nygaard filtration is $\mathcal{N}^{\geq i} = \xi^{(\!(i)\!)}\mathrm{A}_{\mathrm{inf}}$ where $\xi = \phi^{-1}(\widetilde{\xi})$.

Examples (cont.)

Example (q-de Rham prism)

Let $R = \mathbf{Z}_p[\zeta_{p^\infty}]^\wedge_p$. Then $\mathrm{A_{inf}}(R) = \mathbf{Z}_p[q^{1/p^\infty}]^\wedge_{(p,q-1)}$ with $\widetilde{\xi}$ given by the q-**analogue** $(p)_q = 1 + q + \cdots + q^{p-1}$. We have $I_n = (p^n)_q \mathrm{A_{inf}}$ and $\xi = (p)_{q^{1/p}}$.

Example (A non-perfectoid qrsp ring)

Let $S = R[x^{1/p^\infty}]^\wedge_p/(x)$ where R is as above. The prismatic cohomology is a ring of q-divided powers:

$$\Delta_S = \widehat{\bigoplus_{\nu \in \mathbf{N}[1/p]}} \frac{x^{\nu/p}}{\lfloor \nu \rfloor_q !}$$

The Nygaard filtration is $\mathcal{N}^{\geq i} = \widehat{\bigoplus_{\nu}} \xi^{((i-\lfloor \nu \rfloor))} \frac{x^{\nu/p}}{\lfloor \nu \rfloor_q !}$.

Mackey structure

Say $I = (\widetilde{\xi})$ is oriented

$$\phi(\widetilde{\xi}) = \widetilde{\xi}^p + p\delta(\widetilde{\xi})$$

prism condition $\iff$ $\delta(\widetilde{\xi})$ a unit

$$\begin{tikzcd} A/I_2 \arrow[d, bend right, "1"'] \\ A/I \arrow[u, bend right, "\phi(\xi)/\delta(\xi)"'] \end{tikzcd}$$

Mackey condition $FV = p$ becomes

$$\phi(\widetilde{\xi})/\delta(\widetilde{\xi}) \equiv p \bmod I$$

extends to Tambara structure [Prisms and Tambara functors I]

Breuil-Kisin twists

similarly,

$$A/I_n\{1\} = I_n/I_n^2 \to I_{n-1}/I_{n-1}^2 = A/I_{n-1}\{1\}$$

is divisible by p

transversal approximation $\Rightarrow$ **canonically** so

$$A\{1\} := \varprojlim \left(\cdots \xrightarrow{1/p} I_3/I_3^2 \xrightarrow{1/p} I_2/I_2^2 \xrightarrow{1/p} I/I^2 \right)$$

Heuristically:

- $A\{1\} = \bigotimes_{i=0}^{\infty} \phi^i(I)$
- over q-dR prism, $A\{1\} =$ "$(p^\infty)_q A$"

BMS filtration

Theorem (Bhatt-Morrow-Scholze)

If R is a qrsp ring, then

$$\mathrm{TC}^-_{2i}(R;\mathbf{Z}_p) = \mathcal{N}^{\geq i}\hat{\mathbb{\Delta}}_R\{i\}$$
$$\mathrm{TP}_{2i}(R;\mathbf{Z}_p) = \hat{\mathbb{\Delta}}_R\{i\}$$

Lemma (Riggenbach)

We also have

$$\pi_{2i}\mathrm{THH}(R;\mathbf{Z}_p)^{hC_{p^n}} = \frac{\mathcal{N}^{\geq i}\hat{\mathbb{\Delta}}_R\{i\}}{\mathcal{N}^{\geq i+1}\hat{\mathbb{\Delta}}_R\{i\}\cdot I_{n-1}}$$

$$\pi_{2i}\mathrm{THH}(R;\mathbf{Z}_p)^{tC_{p^n}} = \frac{\hat{\mathbb{\Delta}}_R\{i\}}{I_{n-1}}$$

Equivariant Euler classes

V actual representation

the **Euler class**

$$S^0 \xrightarrow{a_V} S^V$$

is given by suspending the inclusion of the zero subspace

gives a class

$$a_V \in \pi^G_{-V} \mathbf{S}_G$$

Equivariant Euler classes

$$S^0 \xrightarrow{a_V} S^V$$

Properties

- $a_V = 0 \iff V^G \neq 0$
- Corollary: if $V^H \neq 0$, then $\mathrm{res}^G_H a_V = 0$
- Corollary: if $V^H \neq 0$, then $a_V \,\mathrm{tr}^G_H(x) = 0$

Let

$$\mathcal{F}_V = \{H \leq G \mid V^H \neq 0\}$$

then "a_V is killed by restriction to $\mathcal{F}_V$, and kills transfers from $\mathcal{F}_V$"

Polar structure of RO(G)

Claim

The multiplication on RO(G) is a red herring: the notion of RO(G)-graded homotopy only uses the additive structure.

Rating: **PARTIALLY FALSE**

$$\begin{tikzcd} S^0 \arrow[r, "a_V"'] \arrow[rr, bend left, "a_{V^{\otimes n}}"] & S^V \arrow[r, dotted, "(-)^{\otimes n}"'] & S^{V^{\otimes n}} \end{tikzcd}$$

$\implies a_V \mid a_{V^{\otimes n}}$. So sees RO(G) as a **polar ring:** abelian group with notion of powers, but not full multiplication (Tilman Bauer: p-polar rings)

Conjecture (S.)

Cotangent cpx for "G-polar rings" (= Mackey functors with norm maps but not full multiplication) should give q-de Rham complex.

Isotropy separation

Let $\mathcal{F}$ be a family of subgroups of G (closed under subgroups and conjugacy)

Universal spaces

$$E\mathcal{F}_+ \to S^0 \to \widetilde{E\mathcal{F}}$$

characterized by

$$(E\mathcal{F})^H = \begin{cases} * & H \in \mathcal{F} \\ \emptyset & H \notin \mathcal{F} \end{cases} \qquad (\widetilde{E\mathcal{F}})^H = \begin{cases} * & H \in \mathcal{F} \\ S^0 & H \notin \mathcal{F} \end{cases}$$

$$S^{\infty V} = \varinjlim (S^0 \xrightarrow{a_V} S^V \xrightarrow{a_V} S^{2V} \xrightarrow{a_V} \cdots) =: S[a_V^{\pm}]$$

is a model for $\widetilde{E\mathcal{F}_V}$!

Isotropy separation = arithmetic square

Isotropy separation square

$$\begin{tikzcd}
(E\mathcal{F}_V)_+ \wedge X \arrow[r] \arrow[d, equal] & X \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \widetilde{E\mathcal{F}_V} \wedge X \arrow[d] \\
(E\mathcal{F}_V)_+ \wedge X \arrow[r] & X^{(E\mathcal{F}_V)_+} \arrow[r] & \widetilde{E\mathcal{F}_V} \wedge X^{(E\mathcal{F}_V)_+}
\end{tikzcd}$$

is an *arithmetic square*

$$\begin{tikzcd}
X_{a_V^\infty\text{-tors}} \arrow[r] \arrow[d, equal] & X \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X[a_V^\pm] \arrow[d] \\
X_{a_V^\infty\text{-tors}} \arrow[r] & X^\wedge_{a_V} \arrow[r] & X^\wedge_{a_V}[a_V^\pm]
\end{tikzcd}$$

Thom classes

In many cases, there exist **Thom classes**

$$u_V \in \pi^G_{|V|-V} X$$

which are invertible in the Borel-completion of X.

Riggenbach (K-theory of double points paper): if $X \in \mathrm{Sp}^{B\mathbf{T}}$ is even, then $u^\pm_V$ exists for X

Corollary: if R is qrsp, then

$$\begin{aligned} \mathrm{TC}^-_\alpha(R; \mathbf{Z}_p) &\cong \mathrm{TC}^-_{2\dim_{\mathbf{C}}(\alpha)}(R; \mathbf{Z}_p) \\ \mathrm{TP}_\alpha(R; \mathbf{Z}_p) &\cong \mathrm{TP}_{2\dim_{\mathbf{C}}(\alpha)}(R; \mathbf{Z}_p) \\ &\cong \mathrm{TP}_0(R; \mathbf{Z}_p) \end{aligned}$$

So... what are we doing?!

Canonicity

The Thom classes u_V are *non-canonical*; want *canonical* descriptions.

What does "canonical" mean?

If $\alpha = i + U - V$ with $i \in \mathbf{Z}$, $U^G = V^G = 0$, then have a *span*

$$\begin{tikzcd} & \underline{\pi}_{i+U}(X) \arrow[dl, "a_U"'] \arrow[dr, "a_V"] & \\ \underline{\pi}_i(X) & & \underline{\pi}_{i+U-V}(X) \end{tikzcd}$$

Identify $\underline{\pi}_\bigstar(X)$ as elements of the span category (in terms of filtrations + fractional ideals)

Canonicity

Hard to distinguish $p\mathbf{Z}_p$ from $\mathbf{Z}_p$ in Ab
But

$$\mathbf{Z}_p := (\mathbf{Z}_p = \mathbf{Z}_p = \mathbf{Z}_p)$$
$$p\mathbf{Z}_p := (\mathbf{Z}_p \hookleftarrow p\mathbf{Z}_p = p\mathbf{Z}_p)$$

are not isomorphic as spans
We do have

$$(\mathbf{Z}_p \xleftarrow{p} \mathbf{Z}_p = \mathbf{Z}_p) \cong (\mathbf{Z}_p \hookleftarrow p\mathbf{Z}_p = p\mathbf{Z}_p)$$

but I had to **label** the map p...
There are sooooooooo many maps between RO(G)-graded groups!

Key idea

By expressing the answer by tensoring with fractional ideals, all maps are "the obvious ones".

Canonicity example

Let A be a ring, $I \subset A$ a principal ideal generated by a non-zerodivisor

Let $\bar{A} = A/I$, let $\bar{A}\{1\} = I/I^2$, and let $M\{n\} = M \otimes_{\bar{A}} (\bar{A}\{1\})^{\otimes n}$ for any $\bar{A}$-module M

If we choose a generator $I = (f)$ and resolve $\bar{A}$ as $(A \xrightarrow{f} A)$, then

$$\mathrm{Tor}^1_A(\bar{A}, M) \cong M_{I\text{-tors}} \qquad \mathrm{Ext}^1_A(\bar{A}, M) \cong M/I$$

But a **canonical** resolution is $(I \to A)$, giving

$$\mathrm{Tor}^1_A(\bar{A}, M) = M_{I\text{-tors}}\{1\} \qquad \mathrm{Ext}^1_A(\bar{A}, M) = M/I\{-1\}$$

Important in Mao's prismatic logarithm paper

Calculations

Fix a qrsp ring R and abbreviate $\mathrm{TC}^- = \mathrm{TC}^-(R; \mathbf{Z}_p)$ etc. Write λ for the obvious representation of $\mathbf{T}$, and $\lambda_n = \lambda^{p^n}$, $\lambda_\infty = \lambda^0$.

Key tool: cell structure

$$\mathbf{T}/C_{p^n} \to S^0 \to S^{\lambda_n}$$

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{TC}^-_{\lambda+2i} \arrow[r, "a_{\lambda_n}"] \arrow[d, equal] & \mathrm{TC}^-_{2i} \arrow[r] \arrow[d, equal] & \mathrm{THH}_{2i}^{hC_{p^n}} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \boxed{I_{n-1}\mathcal{N}^{\geq i+1}\hat{\mathbb{\Delta}}\{i\}} \arrow[r] & \mathcal{N}^{\geq i}\hat{\mathbb{\Delta}}\{i\} \arrow[r] & \dfrac{\mathcal{N}^{\geq i}\hat{\mathbb{\Delta}}}{I_{n-1}\mathcal{N}^{\geq i+1}\hat{\mathbb{\Delta}}} \arrow[r] & 0
\end{tikzcd}$$

Calculations (cont.)

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{TC}^-_{2i} \arrow[r, "a_{\lambda_n}"] \arrow[d, equal] & \mathrm{TC}^-_{2i-\lambda_n} \arrow[r] \arrow[d, equal] & \mathrm{THH}^{hC_{p^n}}_{2(i-1)} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{N}^{\geq i}\hat{\mathbb{\Delta}}\{i\} \arrow[r] & \boxed{\mathcal{N}^{\geq i-1}\hat{\mathbb{\Delta}}\{i\} \otimes I_{n-1}^{-1}} \arrow[r] & \dfrac{\mathcal{N}^{\geq i-1}\hat{\mathbb{\Delta}}}{I_{n-1}\mathcal{N}^{\geq i}\hat{\mathbb{\Delta}}} \arrow[r] & 0
\end{tikzcd}$$

for $k \leq n$:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{TC}^-_{2i+\lambda_k-\lambda_n} \arrow[r, "a_{\lambda_k}"] \arrow[d, equal] & \mathrm{TC}^-_{2i-\lambda_n} \arrow[r] \arrow[d, equal] & \mathrm{THH}^{hC_{p^k}}_{2(i-1)} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \boxed{I_k \otimes I_n^{-1} \otimes \mathcal{N}^{\geq i}\hat{\mathbb{\Delta}}\{i\}} \arrow[r] & I_n^{-1}\mathcal{N}^{\geq i-1}\hat{\mathbb{\Delta}}\{i\} \arrow[r] & \dfrac{\mathcal{N}^{\geq i-1}\hat{\mathbb{\Delta}}\{i-1\}}{I_k \otimes \mathcal{N}^{\geq i}\hat{\mathbb{\Delta}}\{i-1\}} \arrow[r] & 0.
\end{tikzcd}$$

Result

Theorem (S.)

Let R be a qrsp ring with associated prism, and let

$$\alpha = k_0\lambda_0 + \cdots + k_n\lambda_n + k_\infty\lambda_\infty$$

be a virtual C_{p^∞}-representation. Set $d_i = \dim_{\mathbf{C}} \alpha^{C_{p^i}}$.
Then there are **canonical** *identifications*

$$\mathrm{TC}^-_\alpha(R; \mathbf{Z}_p) = \left(\prod_{i=1}^n I_i^{k_i}\right) \mathcal{N}^{\geq d_0} \hat{\mathbb{\Delta}}_R\{k_\infty\}$$
$$\mathrm{TP}_\alpha(R; \mathbf{Z}_p) = \left(\prod_{i=1}^n I_i^{k_i}\right) \hat{\mathbb{\Delta}}_R\{k_\infty\}$$

If we set $I^\alpha = \bigotimes_{i=0}^\infty \phi^i(I)^{d_i}$ and view $\hat{\mathbb{\Delta}}_R\{1\} =: \bigotimes_{i=0}^\infty \phi^i(I)$, then

$$\mathrm{TC}^-_\alpha(R; \mathbf{Z}_p) = \mathcal{N}^{\geq d_0} \hat{\mathbb{\Delta}}_R \otimes I^{\alpha'} \qquad \mathrm{TP}_\alpha(R; \mathbf{Z}_p) = \hat{\mathbb{\Delta}}_R \otimes I^{\alpha'}$$

Even filtration

More generally, let $E \in \mathrm{CAlg}^{B\mathbf{T}}$ be even. Write $A\{i\} := \pi_{2i} E^{t\mathbf{T}}$ with filtration $\mathrm{Fil}^\bullet A\{i\}$ coming from the Tate SS.

Lemma (Antieau-Riggenbach)

Let $[n]_{\mathbf{G}}(t) \in \pi_{-2} E^{h\mathbf{T}} = A\{-1\}$ be the n-series of the FGL of E, and $\langle n \rangle_{\mathbf{G}} = [n]_{\mathbf{G}}/[1]_{\mathbf{G}} \in A$.

$$\pi_{2i} E^{h\mathbf{T}} = \mathrm{Fil}^i A\{i\} \qquad \pi_{2i} E^{t\mathbf{T}} = A\{i\}$$

$$\pi_{2i} E^{hC_n} = \frac{\mathrm{Fil}^i A\{i\}}{\langle n \rangle_{\mathbf{G}} \mathrm{Fil}^{i+1} A\{i\}} \qquad \pi_{2i} E^{tC_n} = \frac{A\{i\}}{\langle n \rangle_{\mathbf{G}} A\{i\}}$$

Write $[n]_{\mathbf{G}}(t) = \prod_{d|n} \Phi_{d,\mathbf{G}}(t)$ (= cyclotomic polynomials for $\widehat{\mathbf{G}}_m$).

Even filtration (cont.)

Theorem (S.)

Let α be a virtual complex $\mathbf{T}$-representation with irreducible decomposition $\alpha = \sum_{i=0}^{\infty} k_i \lambda^i$ and dimension sequence $d_n = \dim_{\mathbf{C}} \alpha^{C_n}$, $d_\infty = \dim_{\mathbf{C}} \alpha^{\mathbf{T}}$.
Then $E_\bigstar^{h\mathbf{T}}$ and $E_\bigstar^{t\mathbf{T}}$ are given **canonically** *by*

$$\begin{aligned} E_\alpha^{h\mathbf{T}} &= \mathrm{Fil}^{d_1} A\{k_0\} \otimes_A \bigotimes_{n>1} (\langle n \rangle_{\mathbf{G}} A)^{\otimes k_n} \\ &= \mathrm{Fil}^{d_1} A \otimes_A \bigotimes_{n>1} (\Phi_{n,\mathbf{G}} A)^{\otimes d_n} \\ E_\alpha^{t\mathbf{T}} &= A\{k_0\} \otimes_A \bigotimes_{n>1} (\langle n \rangle_{\mathbf{G}} A)^{\otimes k_n} \\ &= \bigotimes_{n>1} (\Phi_{n,\mathbf{G}} A)^{\otimes d_n} \end{aligned}$$

Higher topological negative cyclic homology

This is very elegant... but also very boring.
We need to go more genuine!
Set $\mathrm{TC}^{-1} = (\mathrm{TR}^1)^{h\mathbf{T}/C_p}$, equivalently

$$\begin{tikzcd} \mathrm{TC}^{-1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{TC}^- \arrow[d, "\varphi"] \\ \mathrm{TC}^- \arrow[r, "\mathrm{can}"'] & \mathrm{TP} \end{tikzcd}$$

Equivariantly:

$$\mathrm{TC}^{-,\mathrm{can}} = \mathrm{THH}^{h\mathbf{T}}$$
$$\mathrm{TC}^{-,\varphi} = (\mathrm{THH}^{\Phi C_p})^{h\mathbf{T}/C_p}$$

Hesselholt's theorem

R perfectoid ring, $\xi = \phi^{-1}(\widetilde{\xi})$, $\xi_n := \xi\phi^{-1}(\xi)\cdots\phi^{-(n-1)}(\xi)$
q-de Rham prism: $\xi = (p)_{q^{1/p}}$, $\xi_n = (p^n)_{q^{1/p^n}}$
Let $A = \mathrm{A}_{\mathrm{inf}}(R)$, then $A/\xi_n \cong \mathrm{W}_{n-1}(R)$.

Theorem (Hesselholt)

Let S be formally smooth over a perfectoid ring R. Then

$$\mathrm{TR}^n_*(S;\mathbf{Z}_p) = \mathrm{W}_n\Omega^*_{S/R}[\sigma_n]$$

with $F(x) = \phi(x)$, $V(x) = \xi\phi^{-1}(x)$, $R(\sigma_n) = \phi^{-n}(\xi)\sigma_{n-1}$.

Reformulating Hesselholt's theorem

Let $\mathrm{TR}^n_{\alpha,j} = \mathsf{W}\Omega^j$ contribution to $\mathrm{TR}^n_{\alpha+j}$

$$\mathrm{TR}^n_*(S;\mathbf{Z}_p) = \mathsf{W}_n\Omega^*_{S/R}[\sigma_n]$$

$$\mathrm{TR}^n_{2i,j} = \mathsf{W}_n\Omega^j_{S/R}\cdot\sigma_n^i$$

To understand $R(\sigma_n) = \phi^{-n}(\xi)\sigma$:

$$= \mathsf{W}_n\Omega^j_{S/R}\otimes\xi_{n+1}$$

Cartier isomorphism

Say $R = \mathbf{Z}_p[\zeta_{p^\infty}]^\wedge_p$, $S = R[x^\pm]^\wedge_p$
Then Δ_S is a q-de Rham complex

$$\widehat{\bigoplus_{n\in\mathbf{Z}}}\left(A\left\langle x^n\right\rangle \xrightarrow{\nabla_q} A\left\langle x^n \operatorname{dlog} x\right\rangle\right)$$

with $\nabla_q(x^n) = (n)_q x^n \operatorname{dlog} x$

$$\begin{aligned}
\mathrm{H}^0(\Delta_S/\widetilde{\xi}) &= \bigoplus_n A/\widetilde{\xi}\left\langle x^{pn}\right\rangle \\
&\xleftarrow[\sim]{\phi} A/\xi\left\langle x^n\right\rangle \\
\mathrm{H}^1(\Delta_S/\widetilde{\xi}) &= \bigoplus_n A/\ \xi\left\langle x^{pn} \operatorname{dlog} x\right\rangle \\
&\xleftarrow[\sim]{\phi} A/\xi\left\langle x^n \operatorname{dlog} x\right\rangle \otimes \{\xi^{-1}\}
\end{aligned}$$

HOTRSS

If $\alpha \in \mathrm{RO}(\mathbf{Z}/p^n)$, set $\underline{d}_i = \dim_{\mathbf{C}} \alpha^{p^i\mathbf{Z}/p^n\mathbf{Z}}$.
HOTRSS: https://ysulyma.github.io/papers/poly/

Theorem (Contiguous range of $\mathrm{TR}_\bigstar$)

Let $\underline{\alpha} = (\underline{d}_0, \ldots, \underline{d}_n) \in \mathrm{RO}(\mathbf{Z}/p^n)$ *which is* **contiguous:** $\exists\ k \leq \ell$ *such that* $\underline{d}_i \geq 0 \iff k \leq i \leq \ell$.
If S *is smooth over a perfect* $\mathbf{F}_p$*-algebra* k*, then*

$$\mathrm{TR}^n_{\underline{\alpha},j}(S) = \mathrm{Fil}^{k-(\underline{d}_k+\cdots+\underline{d}_\ell)}\, \mathrm{W}_\ell\Omega^j_S \otimes \{p^{\underline{d}_0+\cdots+\underline{d}_\ell}\}$$

$$\mathrm{TR}^n_{\underline{\alpha}-1,j}(S) = \frac{\mathrm{W}_{k-1}\Omega^j_S}{p^{\underline{d}_k+\cdots+\underline{d}_\ell}} \otimes \{p^{\underline{d}_0+\cdots+\underline{d}_{(k-1)}}\}$$

If S *is formally smooth over a* p*-torsionfree perfectoid ring* R^+*,*

$$\mathrm{TR}^n_{\underline{\alpha},j}(S;\mathbf{Z}_p) = \mathrm{Fil}^k\, \mathrm{W}_\ell\Omega^j_{S/R^+} \otimes \bigotimes_{i=0}^{\ell} \{\phi^{-i}(\xi)\}^{\underline{d}_i}$$

$$\mathrm{TR}^n_{\underline{\alpha}-1,j}(S;\mathbf{Z}_p) = \frac{\mathrm{W}_{k-1}\Omega^j_{S/R^+}}{\phi^{-k}(\xi)^{\underline{d}_k}\cdots\phi^{-\ell}(\xi)^{\underline{d}_\ell}} \otimes \bigotimes_{i=0}^{k-1} \{\phi^{-i}(\xi)\}^{\underline{d}_i}$$

Thank you!

Prismatic norms

The prismatic norms on $\mathrm{THH}(\mathrm{qrsp}; \mathbf{Z}_p)^{\{h,t\}C_p\bullet}$ are extremely complicated; here is a preview of what is known about them.

$$\mathrm{Ind}_e^{C_p}(\mathbf{C}) = (p)_\lambda = \lambda^0 + \lambda^1 + \cdots + \lambda^{p-1} \cong (p-1)\lambda_0 + \lambda_\infty$$

but this identification messes up the Mackey structure!

R perfectoid $\implies$

$$\begin{tikzcd}
\frac{\xi A}{\xi^2 A} \arrow[r, "N"] \arrow[rr, bend left, "p\phi(\xi)"] \arrow[dr, "\xi^p"'] & \frac{\xi^p \phi(\xi) A}{\xi^{p+1}\phi(\xi)^2 A} \arrow[r] \arrow[d, "\frac{(p-1)!}{p}"] & \frac{\phi(\xi) A}{\phi(\xi)^2 A} \\
& \frac{\xi^p A}{\xi^{p+1} A} &
\end{tikzcd}$$

Breuil-Kisin norms

Wilson's theorem:

$$(p-1)! \equiv -1 \bmod p$$

Definition (Prismatic Wilson constants)

For a prism (A, I), there is a unique functorial unit $\mathrm{Wil}_1^{\mathbb{\Delta}} \in A/I\phi(I)$ such that

$$\mathrm{Wil}_1^{\mathbb{\Delta}} \equiv \begin{cases} (p-1)! & \bmod I \\ -1 & \bmod \phi(I) \end{cases}$$

Proposition

With R perfectoid, we have

$$N_e^{C_p}(\xi) = \frac{\xi^p \phi(\xi)}{\phi^{-1}(\mathrm{Wil}_1^{\mathbb{\Delta}})\delta(\xi)}$$

More generally, we have the following very mysterious formula.

Definition

An element $\xi \in \mathcal{N}^{\geq 1}\Delta_S$ is **diffractive** if $\delta(\xi) \equiv 1 \bmod \mathcal{N}^{\geq p}\Delta_S$. We set $\widetilde{\xi} := \phi(\xi)/\delta(\xi)$ and $f^{\widetilde{\xi}} := \phi(f) - \widetilde{\xi}\delta(f)$.

Theorem

Let S be qrsp, let $\xi \in \mathcal{N}^{\geq 1}\Delta_S$ be diffractive. Then the norm

$$N^1_{0,1} : \frac{\mathcal{N}^{\geq 1}\hat{\Delta}_R}{\mathcal{N}^{\geq 2}\hat{\Delta}_R} \longrightarrow \frac{\mathcal{N}^{\geq p}(p)_\Delta!\hat{\Delta}_R}{\mathcal{N}^{\geq p+1}I(p)_\Delta!\hat{\Delta}_R} \cong \frac{\mathcal{N}^{\geq p}\hat{\Delta}_R \otimes I}{\mathcal{N}^{\geq p+1}\hat{\Delta}_R \otimes I^2}$$

is given by

$$\begin{aligned} \phi^{-1}\left(\mathrm{Wil}_1^{\Delta}\right) N^1_{0,1}(f) &= \left[(\widetilde{\xi} - p)\phi(f) + \widetilde{\xi} f^{\widetilde{\xi}}\right] \\ &= \left[\widetilde{\xi} f^p + (\widetilde{\xi} - p) f^{\widetilde{\xi}}\right] \\ &= \left[-\phi\psi^p(f) + \widetilde{\xi}(\phi(f) + f^{\widetilde{\xi}})\right] \end{aligned}$$